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FROBENIUS ALGEBRAS OF STANLEY-REISNER RINGS AND MAXIMAL FREE PAIRS

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Abstract. It is known that the Frobenius algebra of the injective hull of the residue field of a formal power series ring modulo a squarefree monomial ideal can be only principally generated or infinitely generated as algebra over its degree zero piece, and that this fact can be read off in the corresponding simplicial complex; in the infinite case, we exhibit a 1–1 correspondence between potential new generators appearing on each graded piece and certain pairs of faces of such a simplicial complex, that we call maximal free pairs.

Introduction

Let Δ be a simplicial complex with n vertices, we say that a pair (F, G) of non–empty, disjoint faces of Δ is **free** provided $F \cup G$ is the intersection of all the facets containing F. Moreover, given two free pairs $(F, G), (F', G')$, we say that $(F, G) \leq (F', G')$ if $F \supseteq F'$ and $G \subseteq G'$; with this partial order, the set of all the free pairs becomes a partially ordered set. In this way, a free pair (F, G) is said to be **maximal** if it is maximal in the set of free pairs with this order relation.

On the other hand, let K be a field, let $I \subseteq K[x_1, \ldots, x_n] = R$ be the squarefree monomial ideal attached to Δ through the Stanley correspondence, and denote by $I^{[2]}$ the ideal obtained after raising to the square all the elements of I; finally, denote by J_1 the smallest ideal of R containing the set $(I^{[2]} :_R I) \setminus (I^{[2]} + (x_1 \cdots x_n))$.

Keeping in mind all the above notations, the main result of this paper (see Theorem 1.1) is the below:

Theorem 0.1. *There is a 1–1 correspondence between the set of minimal monomial generators of J_1 and the set of maximal free pairs of Δ; in particular, the number of maximal free pairs of Δ coincides with the number of minimal monomial generators for J_1.*

Our motivation to obtain this result comes from [ÀMBZ12], where the authors focused on the so–called Frobenius algebras of Stanley–Reisner rings; for the convenience of the reader, in what follows we review some information about these algebras.

Let A be a commutative Noetherian ring of prime characteristic p and let M be an A–module; given any integer $e \geq 1$, we say that a map $M \xrightarrow{\phi} M$ is *p^e-linear* if, for any $a \in A$

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and $m \in M$, $\phi(am) = a^{p^e}\phi(m)$. Since the composition of a p^e-linear map with a $p^{e'}$-linear map produces a $p^{e+e'}$-linear map, one can cook up the algebra

$$\mathcal{F}^M := \bigoplus_{e \geq 0} \operatorname{End}_{p^e}(M),$$

where $\operatorname{End}_{p^e}(M)$ denotes the abelian group made up by all the p^e-linear maps on M; the reader will easily note that $\mathcal{F}^M$ is an associative, positively $\mathbb{N}$–graded, non–commutative ring, and that its degree 1 piece is $\operatorname{End}_A(M)$. $\mathcal{F}^M$ is the so–called *algebra of Frobenius operators* of M. Building upon a counterexample due to Katzman [Kat10], originally motivated by a question raised by Lyubeznik and Smith in [LS01], in [ÀMBZ12] the authors studied $\mathcal{F}^{E_A}$, where $A := K[\![x_1, \ldots, x_n]\!]/I$, K is any field of prime characteristic p, I is a squarefree monomial ideal, and E_A denotes the injective hull of K as A-module; more precisely, it was proved in [ÀMBZ12, Theorem 3.5 and Remarks 3.1.2] that $\mathcal{F}^{E_A}$ is principally generated as A–algebra if and only if $(I^{[2]} : I) = I^{[2]} + (x_1 \cdots x_n)$, otherwise it is infinitely generated as A–algebra. One question not answered in [ÀMBZ12] was whether it is possible to read off the principal (respectively, the infinite) generation of $\mathcal{F}^{E_A}$ in the simplicial complex Δ associated to I through the Stanley correspondence; this question was answered in [ÀMY14], where Àlvarez Montaner and Yanagawa show that, if $\Delta = \operatorname{core}(\Delta)$, then $\mathcal{F}^{E_A}$ is principally generated if and only if Δ does not have free faces (equivalently, if and only if any proper face is contained in at least two facets).

Hereafter, suppose that $\Delta = \operatorname{core}(\Delta)$, and that $\mathcal{F}^{E_A}$ is infinitely generated as A–algebra; on the one hand, by [ÀMBZ12, Theorem 3.5], one knows that $\mathcal{F}^{E_A}$ is infinitely generated if and only if $(I^{[2]} : I) = I^{[2]} + J_1 + (x_1 \cdots x_n)$, where $0 \neq J_1 \not\subseteq I^{[2]} + (x_1 \cdots x_n)$ is the smallest monomial ideal containing the set $(I^{[2]} : I) \setminus I^{[2]} + (x_1 \cdots x_n)$. On the other hand, by [ÀMY14] $\mathcal{F}^{E_A}$ is infinitely generated if and only if Δ has at least one free face. Keeping in mind these two characterizations, one can ask the following:

Question 0.2. *Is there some kind of relation between the number of minimal monomial generators of J_1 and the number of free faces of Δ?*

As we will see, such a relation exists but not directly with the free faces of Δ, instead with maximal free pairs, whereas it is easy to see that free faces are minimal elements of this finite poset. In fact, the correspondence given in Theorem 0.1 is explicit and one can easily extract from the maximal free pairs the corresponding minimal monomial generators of J_1. As application, we use Theorem 0.1 to show (see Theorem 2.1) that, when $\mathcal{F}^{E_A}$ is infinitely generated as A–algebra, the number of new generators appearing on each graded piece is always less or equal than the number of maximal free pairs of the simplicial complex Δ.

The content of this paper is based on [BZ], where the reader can find all the details.

1. Main result

In what follows, let K be a field, and $R = K[x_1, \ldots, x_n]$; we abbreviate the set $\{1, \ldots, n\}$ writing just $[n]$. Finally, given a monomial $m \in R$ we denote by $\operatorname{supp}(m)$ its support and by $\operatorname{supp}_2(m)$ the set of indices i such that $\deg_{x_i}(m) = 2$; keeping in mind these notations, the main result of this paper (see [BZ, Theorem 2.15] for the proof) is the below:

Theorem 1.1. *On the one hand, given a free pair* (F, G) *as above, set*

$$A(F, G) := \left(\prod_{i \in F} x_i^2\right)\left(\prod_{i \notin F \cup G} x_i\right).$$

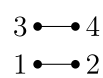
On the other hand, given a monomial $m \in R$ *set* $Y(m) := (\mathrm{supp}_2(m), [n] \setminus \mathrm{supp}(m))$. *Then, the set* $\{A(F, G) :\ (F, G)$ *maximal free pair of* $\Delta\}$ *is the minimal monomial generating set for* J_1. *Moreover,* A *and* Y *define* $1-1$ *correspondences between the set of maximal free pairs of* Δ *and the set of minimal monomial generators of* J_1.

One can turn Theorem 1.1 into a naive algorithm which, receiving any simplicial complex Δ with n vertices as input, returns all its non–empty maximal free pairs in a list L; this method works in the below way.

(i) Compute the corresponding Stanley–Reisner ideal $I \subseteq K[x_1, \dots, x_n]$, where K is any field.
(ii) Compute $J := (I^{[2]} : I)/(I^{[2]} + (x_1 \dots x_n))$.
(iii) If $J = 0$, then output that Δ has no free pairs and stop.
(iv) Otherwise, for each minimal monomial generator m of J, add to L the pair (F, G), where $F := \{i \in [n] :\ \deg_{x_i}(m) = 2\}$ and $G := \{i \in [n] :\ \deg_{x_i}(m) = 0\}$.
(v) Output the list L.

We have already implemented this algorithm in Macaulay2 (see [GS13] and [BZ16]); next, we show two examples where we explain, on the one hand, how to use our implementation and how to interpret the output, and, on the other hand, why we really need to consider maximal free pairs, and not just free faces.

Example 1.2. Let Δ be the simplicial complex depicted below:

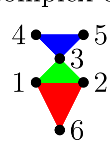


It is easy to see that, for each vertex, there is exactly a maximal free pair and these are all. We check this by using our implementation in the below way:

```
clearAll;
load "FreePairs.m2";
R=ZZ/2 [x,y,z,w,Degrees=>entries id_(ZZ^4)];
I=ideal(x*z,x*w,y*z,y*w);
L=freePairs(I);
L
{{{4}, {3}}, {{2}, {1}}, {{1}, {2}}, {{3}, {4}}}
```

Indeed, the Stanley–Reisner ideal is $I := (xz, xw, yz, yw) \subseteq K[x, y, z, w]$, and $(I^{[2]} : I) = I^{[2]} + (x^2zw, xyz^2, xyw^2, y^2zw) + (xyzw)$, this shows that the maximal free pairs (**that are also the free faces**) of Δ are $(\{1\}, \{2\})$, $(\{3\}, \{4\})$, $(\{4\}, \{3\})$ and $(\{2\}, \{1\})$.

Example 1.3. Let Δ be the simplicial complex depicted below:



We use our method to determine all the possible maximal free pairs of Δ as follows:

```
clearAll;
load "FreePairs.m2";
R=ZZ/2 [x,y,z,w,a,b,Degrees=>entries id_(ZZ^6)];
I=ideal(x*w,x*a,y*w,y*a,z*b,w*b,a*b);
L=freePairs(I);
L
{{{6}, {1, 2}}, {{5}, {3, 4}}, {{4}, {3, 5}}, {{2}, {1}},
{{1}, {2}}}
```

Notice that, when (F, G) is either $(\{1\}, \{2\})$ or $(\{2\}, \{1\})$, the face $F \cup G$ turns out to be the intersection of facets $\{1, 2, 3\}$ and $\{1, 2, 6\}$, which are the facets containing F.

2. Application to Frobenius algebras of Stanley–Reisner rings

As we explained in the Introduction, we use in [BZ] Theorem 1.1 to prove the following interesting consequence about these algebras; this is the last result of this paper.

Theorem 2.1. *Let K be any field of prime characteristic, let $R := K[\![x_1, \dots, x_n]\!]$, let $I = I_\Delta \subseteq R$ be a squarefree monomial ideal, let $A := R/I$, and let E_A denote the injective hull of K as A-module. If $\mathcal{F}^{E_A}$ is infinitely generated as A-algebra, then the number of new generators appearing on each graded piece is always less or equal than the number of maximal free pairs of the simplicial complex Δ.*

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