Monografías de la Real Academia de Ciencias. Zaragoza. **43:** 63–65, (2018).
ISSN: 1132-6360

EQUATIONS FOR THE FLEX LOCUS OF A HYPERSURFACE

LAURENT BUSÉ, CARLOS D'ANDREA, MARTÍN SOMBRA, AND MARTIN WEIMANN

Abstract. For a degree d surface in projective space with no ruled components, a theorem of Salmon asserts that the flex locus is a curve on this surface of degree at most $11d^2 - 24d$. We generalise this result to hypersurfaces of arbitrary dimension and compute explicit equations of the flex locus by using multidimensional resultant theory. For generic hypersurfaces, we show that our degree bound is reached and that the generic flex line is unique and with expected contact order.

The flex locus of a projective hypersurface is the subset of points through which there is a line with higher contact order than expected. For a plane projective curve, the flex locus is well known: it is the set of inflexion points and it is given by the intersection of the curve with the zero locus of the Hessian. In this article, we study the basic geometry of the flex locus. In particular, we address the problem of computing the dimension, degree and equations of the flex locus of a hypersurface of a projective space of arbitrary dimension.

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $V \subset \mathbb{P}^n$ a subvariety of the n-dimensional projective space over $\mathbb{K}$. The *osculating order* of V at a point $p \in \mathbb{P}^n$ is the maximum contact order at p between V and a line. We denote it by $\mu_p(V)$. An irreducible variety is ruled if it is a union of lines or, equivalently, if the osculating order is infinite for all points $p \in V$. In addition, we say that a property holds for a *generic* element in a family if it holds outside a proper Zariski closed subset of this family.

The following result is a consequence of [Lan99, Generic Theorem]:

Theorem 0.1. *Let $V \subset \mathbb{P}^n$ be a hypersurface with no ruled components, and $p \in V$. Then $\mu_p(V) \geq n$ and the equality holds for a generic $p \in V$.*

This result leads to the following definition:

Definition 0.2. We call the *flex locus* of a projective hypersurface $V \subset \mathbb{P}^n$ the subset

$$\mathrm{Flex}(V) = \{p \in V \mid \mu_p(V) > n\}.$$

A point $p \in \mathrm{Flex}(V)$ is called a *flex point.* A line with contact order larger than n at some point of V is called a *flex line.*

Let us first summarize the known results in the case of curves and surfaces.

Theorem 0.3. *Let $C \subset \mathbb{P}^2$ be a plane curve of degree d which does not contain any line. Then C contains at most $3d^2 - 6d$ flex points.*

This result follows directly from the fact that $p \in C$ is a flex point if and only if the Hessian of the homogeneous polynomial equation of C vanishes at p (see for instance [BK86, Thm.1, Ch.7.3]). In the case of surfaces, the following result is due to Salmon [Sal49, Ch.VI] but it has been revisited by several authors since then (e.g. [Kat14, EH13]):

Theorem 0.4. *Let $S \subset \mathbb{P}^3$ be a surface of degree d with no ruled components. Then $Flex(S)$ is a curve on S of degree at most $11d^2 - 24d$.*

In particular, if $d = 3$, we get that $\deg \mathrm{Flex}(S) \leq 27$. Since a flex line of a cubic surface S has contact order ≥ 4, it is necessarily contained in S by Bezout's theorem: we recover the classical fact that a cubic contains at most 27 lines (and it turns out to be the exact number when S is smooth).

Our first main theorem generalises these results to hypersurfaces of arbitrary dimension. Its proof follows the original proof given by Salmon in the case of surfaces and leads to the construction of explicit equations for the flex locus. To be more precise, let us introduce two sets of variables

$$x = (x_0, \ldots, x_n) \quad \text{and} \quad y = (y_0, \ldots, y_n).$$

To any homogeneous polynomial $F \in \mathbb{K}[x]$, we associate the family of homogeneous polynomials $F_0, \ldots, F_d \in \mathbb{K}[x, y]$ which are uniquely determined by the formula

$$F(x + ty) = \sum_{k=0}^{d} F_k(x, y) \frac{t^k}{k!}. \tag{1}$$

For all $k = 0, \ldots, d$, the polynomial F_k is bihomogeneous of bidegree $(d-k, k)$ with respect to x and y and it admits an explicit expression that depends on the k^{th}-order partial derivatives of F. In particular, we have that $F_0(x, y) = F(x)$ and $F_d(x, y) = F(y)$. Below, the resultant operator of $n + 1$ homogeneous polynomials in the $n + 1$ variables $y_0, \ldots, y_n$ is denoted by $\mathrm{Res}_y(\cdot)$.

Theorem 0.5. *Let $V \subset \mathbb{P}^n$ be a hypersurface defined by a square-free polynomial F of degree d. If $d < n$, all components of V are ruled. If $d \geq n$, then either V has a ruled component or $Flex(V)$ is a codimension 1 subvariety of V with equations*

$$Flex(V) = \{F = P = 0\}, \tag{2}$$

where $P \mod F$ is uniquely determined by

$$Res_y(F_1, \ldots, F_n, y_0) \equiv x_0^{n!} P \mod F. \tag{3}$$

Moreover, in this case the flex locus is equipped with a scheme structure and we have that

$$\deg\ Flex(V) = d^2 \sum_{k=1}^{n} \frac{n!}{k} - d(n + 1)! \tag{4}$$

Example 0.6. Let $C = \{F = 0\} \subset \mathbb{P}^2$ be a smooth curve of degree d. A straightforward computation using the Euler identities shows that

$$-(d-1)^2 \mathrm{Res}_y(F_1, F_2, y_0) \equiv x_0^2 \det(H_F) \mod F, \tag{5}$$

where H_F stands for the Hessian of F (determinant of the Hessian matrix): we recover the well–known fact that $p \in C$ is a flex if and only if the Hessian of F vanishes at p.

Our second main result ensures that the degree bound is sharp and that the expected properties of the flex locus hold in the generic case. More precisely :

Theorem 0.7. *For F a generic polynomial of degree $d \geq n$, the subscheme $Z(F, P)$ defined by (2) and (3) is reduced and the degree bound (4) is reached. Moreover, through a generic flex point, there is a unique flex line, and this line has contact order exactly $n+1$ (or is contained in V if $d = n$).*

We conclude by mentioning that giving a closed form for a canonical representative for P modulo F in Theorem 0.5 seems to be a challenge on its own. In the case of curves, such a representative is given by the Hessian. For $n = 3$, Salmon also obtained a representative of this polynomial as a determinantal closed formula in terms of covariants, based on an approach by Clebsch [Sal65, Articles 589 to 597]. It would be interesting to generalize these formulae to higher dimensions.

Acknowledgements. All authors where supported by the CNRS PICS 6381 "Géométrie diophantienne et calcul formel". D'Andrea and Sombra also acknowledge financial support from the Spanish MINECO, through the research project MTM2015-65361-P, and the "María de Maeztu" Programme for Units of Excellence in R&D (MDM-2014-0445).

References

[BK86] E. Brieskorn, H. Knörrer, *Plane algebraic curves*, Birkhäuser Basel, 1986.

[EH13] D. Eisenbud, J. Harris. *3264 and All That; Intersection Theory in Algebraic Geometry.* Cambridge University Press; 1 edition, 2016.

[Kat14] N. Katz. *The flecnode polynomial: a central object in incidence geometry.* Proceedings of the 2014 ICM, arXiv:1404:3412.

[Lan99] J. M. Landsberg. *Is a linear space contained in a submanifold? – On the number of derivatives needed to tell.* J. Reine Angew. Math. 508 (1999), 53–60.

[Sal49] G. Salmon. *On the triple tangent planes to a surface of the third order,* Cambridge and Dublin Math. Journal 4 (1849), 252–260.

Université Côte d'Azur, Inria, 2004 route des Lucioles, 06902 Sophia Antipolis, France
E-mail address: `Laurent.Buse@inria.fr`
URL: `http://www-sop.inria.fr/members/Laurent.Buse/`

Departament de Matemàtiques i Informàtica, Universitat de Barcelona. Gran Via 585, 08007 Barcelona Spain
E-mail address: `cdandrea@ub.edu`
URL: `http://www.ub.edu/arcades/cdandrea.html`

ICREA. Passeig Lluís Companys 23, 08010 Barcelona, Spain & Departament de Matemàtiques i Informàtica, Universitat de Barcelona. Gran Via 585, 08007 Barcelona Spain
E-mail address: `sombra@ub.edu`
URL: `http://www.maia.ub.es/~ sombra`

Université de Caen UMR CNRS 6139 BP 5186 -14032 Caen Cedex, France & GAATI (University of French Polynesia), BP 6570 - 98702 Faaa
E-mail address: `weimann@unicaen.fr`
URL: `https://weimann.users.lmno.cnrs.fr/`