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TWO NEW CHARACTERIZATIONS OF FREE HYPERPLANE ARRANGEMENTS

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Abstract. We describe two new characterizations of freeness for hyperplane arrangements via the study of the generic initial ideal and of the sectional matrix of the Jacobian ideal of arrangements. Moreover, we will show the new package `arrangements` fot the software CoCoA.

1. Introduction

An arrangement of hyperplanes is a finite collection of codimension one affine subspaces in a finite dimensional vector space. Associated to these spaces, there is a plethora of algebraic, combinatorial and topological invariants. Arrangements are easily defined but they lead to deep and beautiful results that put in connection various area of mathematics. We refer to [9] for a comprehensive treatment of the subject.

In the theory of hyperplane arrangements, the freeness of an arrangement is a key notion which connects arrangement theory with algebraic geometry and combinatorics. The notion of freeness was introduced by Saito in [11] for the case of hypersurfaces in the analytic category. The special case of hyperplane arrangements was firstly studied by Terao in [12], where he showed that we can pass from analytic to algebraic considerations. By definition, an arrangement is free if and only if its module of logarithmic derivations is a free module. It turns out that, by Terao's characterization [9], this notion is equivalent to the requirement that the Jacobian ideal of the arrangement (the ideal generated by the defining equation and its partial derivatives) is Cohen-Macaulay of codimension 2. To check freeness for a given arrangement, or to construct new free arrangements, is a very difficult task though it is very fundamental.

We will give new characterizations of freeness for any dimension. Namely, starting from the result of Terao, we characterize freeness in terms of the generic initial ideal and of the sectional matrix of the Jacobian ideal $J(\mathcal{A})$ of the arrangement $\mathcal{A}$, Moreover, we will describe the package `arrangements` that we developed for the software CoCoA.

These results are part of [5] and [10].

2. Preliminares on hyperplane arrangements

Let K be a field of characteristic zero. A finite set of affine hyperplanes $\mathcal{A} = \{H_1, \dots, H_n\}$ in K^l is called a **hyperplane arrangement**. For each hyperplane H_i we fix a defining equation $\alpha_i \in S = K[x_1, \dots, x_l]$ such that $H_i = \alpha_i^{-1}(0)$, and let $Q(\mathcal{A}) = \prod_{i=1}^{n} \alpha_i$. An arrangement $\mathcal{A}$ is called **central** if each H_i contains the origin of K^l.

We denote by $\mathrm{Der}_{K^l} = \{\sum_{i=1}^{l} f_i \partial_{x_i} \mid f_i \in S\}$ the S-module of **polynomial vector fields** on K^l (or S-derivations). Let $\delta = \sum_{i=1}^{l} f_i \partial_{x_i} \in \mathrm{Der}_{K^l}$. Then δ is said to be **homogeneous of polynomial degree** d if $f_1, \dots, f_l$ are homogeneous polynomials of degree d in S. In this case, we write $\mathrm{pdeg}(\delta) = d$.

A central arrangement $\mathcal{A}$ is said to be **free with exponents** $(e_1, \dots, e_l)$ if and only if the module of vector fields logarithmic tangent to $\mathcal{A}$, $D(\mathcal{A}) = \{\delta \in \mathrm{Der}_{K^l} \mid \delta(\alpha_i) \in \langle\alpha_i\rangle S, \forall i\}$, is a free S-module and there exists a basis $\delta_1, \dots, \delta_l \in D(\mathcal{A})$ such that $\mathrm{pdeg}(\delta_i) = e_i$, or equivalently $D(\mathcal{A}) \cong \bigoplus_{i=1}^{l} S(-e_i)$.

3. New characterizations of free hyperplane arrangements

We firstly characterize freeness by looking at the generic initial ideal $\mathrm{rgin}(J(\mathcal{A}))$ of the Jacobian ideal $J(\mathcal{A})$ of $\mathcal{A}$ with respect to the term ordering degrevlex. In this setting, the generic initial ideal of a polynomial ideal I with respect to a term ordering σ is the unique monomial ideal J such that $J = \mathrm{LT}_\sigma(g(I))$, where g is a generic change of coordinates. For the detailed definition and the basic properties of generic initial ideals, we refer to [7] and [8].

Theorem 3.1. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement in K^l. Then $\mathcal{A}$ is free if and only if* $\mathrm{rgin}(J(\mathcal{A}))$ *is S or its minimal generators include x_1^{n-1}, some positive power of x_2, and no monomials in $x_3, \dots, x_l$. More precisely, if $\mathcal{A}$ is free, then* $\mathrm{rgin}(J(\mathcal{A}))$ *is S or it is minimally generated by*

$$x_1^{n-1},\ x_1^{n-2}x_2^{\lambda_1},\ \dots,\ x_2^{\lambda_{n-1}}$$

with $1 \le \lambda_1 < \lambda_2 < \cdots < \lambda_{n-1}$ and $\lambda_{i+1} - \lambda_i = 1$ or 2.

In the example at the end of the next section, we can see that the generic initial ideal of the Jacobian ideal of the Braid arrangement involves only the first two variables, in fact the Braid arrangement is free.

If we look at the resolution of the $\mathrm{rgin}(J(\mathcal{A}))$, we can not only see if $\mathcal{A}$ is free but also compute its exponents.

Theorem 3.2. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential and central arrangement in K^l, with $l \ge 2$. If $\mathcal{A}$ is free with exponents $(e_1, \dots, e_l)$ then* $\mathrm{rgin}(J(\mathcal{A}))$ *has free resolution*

$$0 \longrightarrow \bigoplus_{j=n-1}^{n+e_l-2} S(-j-1)^{\beta_{1,j+1}} \longrightarrow \bigoplus_{j=n-1}^{n+e_l-2} S(-j)^{\beta_{0,j}} \longrightarrow \mathrm{rgin}(J(\mathcal{A})) \longrightarrow 0,$$

where $\beta_{0,n-1} = \beta_{1,n}+1 = l$ and $\beta_{1,j+1} = \beta_{0,j} = \#\{i \mid e_i > j-n+1\}$ for all $j \ge n$. In particular, $\beta_{0,n-1} > \beta_{0,n} \ge \cdots \ge \beta_{0,n+e_l-2}$.

We now characterize freeness by looking at the sectional matrix of $S/J(\mathcal{A})$. In this setting, the sectional matrix $\mathcal{M}_{S/I}$ of a polynomial ideal I encodes the Hilbert functions of successive hyperplane sections of the quotient S/I. In particular, $\mathcal{M}_{S/I}(i, -)$ is the Hilbert function of the quotient $S/(I + (L_1, \dots, L_{l-i}))$, where L_k are generic linear forms. For the detailed definition and basic properties of sectional matrices, we refer to [6] and [4].

Theorem 3.3. *Let $\mathcal{A}$ be a central arrangement and $d_0 = \max\{d \mid \mathcal{M}_{S/J(\mathcal{A})}(2, d) \neq 0\}$. Then $\mathcal{A}$ is free if and only if $\mathcal{M}_{S/J(\mathcal{A})}$ is the zero function or the following two conditions hold*

(1) $\mathcal{M}_{S/J(\mathcal{A})}(3, d_0) = \mathcal{M}_{S/J(\mathcal{A})}(3, d_0+1) = \mathcal{M}_{S/J(\mathcal{A})}(3, d_0 + 2)$,
(2) $\mathcal{M}_{S/J(\mathcal{A})}(3, d_0) = \sum_{d=0}^{d_0} \mathcal{M}_{S/J(\mathcal{A})}(2, d)$.

In the example at the end of the next section, we can see that the sectional matrix of the Jacobian ideal of the Braid arrangement satisfies both the conditions of the previous theorem.

With the notation of the previous theorem, d_0 coincides with $\min\{d \mid x_2^{d+1} \in \mathrm{rgin}(J(\mathcal{A}))\}$.

Conjecture 3.4. *Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a central arrangement in K^l. If $\mathrm{rgin}(J(\mathcal{A}))$ has a minimal generator T that involves the third variable of S, then $\deg(T) \geq d_0 + 1$.*

If the previous conjecture is true, then the statement of Theorem 3.3 becomes easier, as follows:

Corollary 3.5. *Let $\mathcal{A}$ be a central arrangement. Then $\mathcal{A}$ is free if and only if $\mathcal{M}_{S/J(\mathcal{A})}$ is the zero function or $\mathcal{M}_{S/J(\mathcal{A})}(3, d_0) = \mathcal{M}_{S/J(\mathcal{A})}(3, d_0{+}1) = \mathcal{M}_{S/J(\mathcal{A})}(3, d_0 + 2)$.*

4. Arrangement package for CoCoA

In order to test our theorems and play with examples, the second and third authors (see [10]) wrote the package `arrangements` for the software CoCoA, see [1], [2] and [3]. This package will be part of the official release CoCoA-5.2.4.

This package allows the user to easily define any hyperplane arrangement. Moreover, several known families of arrangements are already implemented. For example, we can construct the Braid arrangement in CoCoA as follows:

```
/**/ use S::=QQ[x,y,z];
/**/ A := ArrBraid(S,3);A;
[x-y,  x-z,  y-z]
```

With this package, we can compute several combinatorial invariants of hyperplane arrangements. For example, we can construct the flats of the intersection lattice, the characteristic and the Tutte polynomials, and the Betti numbers of the Braid arrangement in CoCoA as follows:

```
/**/ ArrFlats(A);
[[ideal(0)], [ideal(x-y), ideal(x-z), ideal(y-z)], [ideal(x-z,y-z)]]
/**/ ArrCharPoly(A);
t^3-3*t^2+2*t
/**/ ArrTuttePoly(A);
t[1]^2+t[1]+t[2]
/**/ ArrBettiNumbers(A);
[1,  3,  2]
```

We can also compute various algebraic invariants. For example, we can construct the Orlik-Terao ideal of the Braid arrangement in CoCoA as follows:

```
/**/ OrlikTeraoIdeal(A);
ideal(y[1]*y[2]-y[1]*y[3]+y[2]*y[3])
```

Moreover, several functions for the class of free hyperplane arrangements are implemented. In addition, this package allows also to do computations with multiarrangements. We can check freeness, compute a Saito's matrix, the exponents, the generic initial ideal and the sectional matrix of the Braid arrangement in CoCoA as follows:

```
/**/ IsArrFree(A);
true
```

```
/**/ ArrDerMod(A);
matrix( /*RingWithID(3, "QQ[x,y,z]")*/
 [[1, 0, 0],
  [1, x-y, 0],
  [1, x-z, x*y-x*z-y*z+z^2]])
/**/ ArrExponents(A);
[0,  1,  2]
/**/ Q:=product(A);
/**/ GinJacobian(Q);
ideal(x^2,  x*y,  y^3)
/**/ PrintSectionalMatrix(S/ideal(GensJacobian(Q)));
  0  1  2  3  4
  -  -  -  -  -
  1  1  0  0  0
  1  2  1  0  0
  1  3  4  4  4
```

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